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LETTER TO THE EDITOR

Three properties of the infinite cluster in percolation theory

Alex Hankey†

Maharishi European Research University, CH-6446 Seelisberg, Switzerland

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Abstract. We present proofs of the following properties of the infinite cluster in percolation theory for all $p > p_c$ and for all lattices.

(i) The ratio of the number of boundary sites to cluster sites is $(1-p)/p$.

(ii) The specific logarithmic multiplicity of the infinite cluster per cluster site is equal to the specific logarithmic multiplicity of all configurations of the lattice per occupied site, $\ln p + [(1-p)/p] \ln(1-p)$.

(iii) The specific logarithmic multiplicity of the infinite cluster per cluster site is given by $S(a) = (1+a)\ln(1+a) - a \ln a$ for all $a < a_c$.

We derive a limiting form for the multiplicity of finite clusters for $a < a_c$ and show that the simplest n-dependent term varies as $n^{-1/d}$ where d is the dimensionality of the lattice. We also suggest that for $a > a_c$, $dS(a)/da \leq 0$ and that $S(a_c)$ is the maximum value of $S(a)$ and that its first derivative is discontinuous at $a = a_c$.

In a recent paper Domb (1976) has investigated the statistics of lattice animals in relation to percolation theory, and introduced the ratio $b/n = a$ as a convenient parameter for their classification. Here n is the number of sites in the animal, and b the number of defining boundary sites. Domb focused particular attention on $M(n, b)$ the number of clusters on a given lattice, and its limiting form as $n \to \infty$ for fixed a which we shall write as‡

$$\lim_{\substack{n\to\infty \\ b/n=a}} \left(\frac{1}{n}\right) \ln M(n, b) = S(a). \tag{1}$$

Possibly the most interesting proposal to date about the value of $S(a)$ is that of Leath (1976) who suggested that

$$S(a) = (1+a)\ln(1+a) - a \ln a. \tag{2}$$

The region of applicability of this relation has not been clarified and Domb (1976) has rightly pointed out that this function does not permit the derivation of a clear cut value a_c for critical percolation, and thus not for p_c either.

An important result obtained by Leath and Domb provides a simple relation between a_c and p_c at critical percolation:

$$a_c = (1-p_c)/p_c. \tag{3}$$

† Now at Academy for SCI, Rt. 43, Hancock Road, Williamstown, Mass. 01267, USA.

‡ Here we have used the symbol $S(a)$ rather than Domb's $\ln \Lambda(a)$ to emphasise the fact that this function is closely related to the specific entropy.

References to related literature on percolation can be found in the two papers quoted above.

In this letter we shall show that equations similar to (2) and (3) may be obtained for all infinite clusters for $p > p_c$ (and $a < a_c$) and that they provide direct confirmation of equation (3) and of the way that equation (2) should be applied at percolation. Furthermore, they suggest directly a new approach to the solution of percolation problems on real finite-dimensional lattices.

The three properties we shall prove for the infinite cluster are valid on all finite-dimensional lattices apparently even including those with defects. The first is also valid for pseudolattices; a discussion of the second and third for pseudolattices will be given in a subsequent publication giving a fuller version of the present work.

They are as follows:

(i) The ratio a of the number of boundary sites to the number of cluster sites for the infinite cluster is given by

$$a = (1-p)/p \tag{4}$$

for all values of $p > p_c$, and hence of $a < a_c$. Thus the value of this ratio a decreases smoothly from the value zero at $p = 1$ to its value at criticality given by equation (3). Furthermore, since $b = an$ all naturally occurring infinite clusters have $a > 0$ (except for the trivial case $p = 1$) and are ramified.

(ii) The specific entropy per occupied site of the infinite cluster $S(a)$ is equal to the specific logarithmic multiplicity of all possible configurations for the whole lattice. The implications of this statement would appear to be that there is nothing otherwise special or selected about a site that is a member of the infinite cluster. The infinite cluster is egalitarian.

(iii) The specific entropy of the infinite cluster $S(a)$ depends on the ratio a precisely according to equation (2).

The third proposition is probably of greatest consequence, it can easily be obtained from (i) and (ii) by using equation (4) to substitute a for p in the expression for the specific logarithmic multiplicity of all lattice configurations. We shall later give a simpler analytic proof of these results, but we first give a more concrete discussion which provides greater insight into their significance.

Proof of (i)

Property (i) is very obviously true for $p \simeq 1$. This is because in this limit *all* the unoccupied lattice sites become boundary sites for the infinite cluster, which similarly comprises all occupied sites. Hence it is obvious that equation (4) is true and (i) holds. We can extend this easily enough to cases where all sites that are not part of the infinite cluster form finite size enclosures within the infinite cluster, and the method of so doing will provide the insight to extend this to all $p > p_c$.

Any finite enclosure within the infinite cluster contains the following kinds of site:

(a) defining boundary sites for the infinite cluster that are unoccupied or empty E_b: these may be assigned a probability $q(E_b)$ for occurrence over the entire lattice;

(b) all other empty sites E_0—of probability $q(E_0)$;

(c) all occupied sites that are full (other than those in the infinite cluster) F_0—of probability $p(F_0)$.

Together with infinite cluster sites F_c (of probability $p(F_c)$) these account for the entire lattice. We can therefore subdivide the probabilities p and $q(=1-p)$ as:

$$p=p(F_c)+p(F_0) \tag{5a}$$

$$q=q(E_b)+q(E_0). \tag{5b}$$

(Note that E_0 sites may or may not be boundary sites of finite clusters; but they are *not* boundary sites of the infinite cluster.)

We now regard the set of all enclosures of a given shape as providing a statistical ensemble of lattice sites and calculate the corrections to $p(F_c)$ and $q(E_b)$. On a square lattice the smallest such enclosure has one site that may be occupied or unoccupied, and four defining boundary sites: we may assign it a probability P_1 of being occupied given by:

$$P_1=p(1-p)^4 f_1(p) \tag{6a}$$

and of being unoccupied

$$Q_1=q(1-p)^4 f_1(p) \tag{6b}$$

where $f_1(p)=p(F_c)(p^{15}+\text{higher-order terms})$ ensures that the eight infinite cluster sites defining this enclosure are indeed connected. Setting $p(F_0)\simeq P_1$ and $q(E_0)\simeq Q_1$ as the smallest correction yields with equations (4*a*) and (4*b*),

$$a=\frac{q(E_b)}{p(F_c)}=\frac{q-Q_1}{p-P_1}=\frac{1-p}{p}. \tag{7}$$

This procedure may be repeated for *any* finite enclosure for which equations analogous to equations (6*a*) and (6*b*) may be written. Once the enclosure has been defined and assigned a probability, the occupation of sites *within* the enclosure is random and the ratio of $q(E_b)$ to $p(F_c)$ reduces to $(1-p)/p$ as in (7).

To complete the proof of (i) for all lattices we must take into account the fact that the sites E_0 and F_0 may themselves form infinite clusters. We can circumvent this possibility by using the stratagem of a cyclic boundary (as elsewhere in solid state physics). Instead of an 'infinite cluster' we use the concept of a 'spanning cluster'. We are of course interested in the limit when the size of the torus becomes very large, but as long as it is finite, enclosures which are not part of the spanning cluster are finite, the argument of the previous section applies, and relation (7) is satisfied.

Proof of (ii)

The specific logarithmic multiplicity of all possible configurations on a lattice with site occupancy probability p may be calculated by finding that for a finite sample size of lattice N and letting N become infinite. For such a sample the number of possible configurations is given by $N!/(N_0!\,N_u!)$ where N_0 is the number of occupied sites and N_u the number of unoccupied sites. Taking logarithms, dividing by N, and using the fact that $N_0\simeq pN$ and $N_u\simeq(1-p)N$ exactly in the limit $N\to\infty$ yields

$$s(p)=p\ln p+(1-p)\ln(1-p) \tag{8}$$

where we denote the specific logarithmic multiplicity by $s(p)$ as it is equal to the specific entropy in the non-interacting limit $J/T \to 0$ of the Ising model. The corresponding quantity per *occupied site* is given by

$$S(p) = \frac{1}{p} s(p) = \ln p + \frac{1-p}{p} \ln(1-p). \tag{9}$$

The number of configurations of the entire lattice C can be counted differently however: they are exactly equal to the number of possible configurations C_0 of sites E_0 and F_0 summed over each possible configuration of the infinite cluster (sites F_c and E_b). However since these various kinds of site form fractions of the lattice $q(E_0)$, $p(F_0)$, etc, and because the most probable fractions *dominate completely* so that other possibilities may be neglected one can derive the multiplication rule in the limit of very large sample size N of the occupied lattice:

$$C = C_0 C_\infty \tag{10}$$

where C_∞ is the number of configurations of the infinite cluster. Therefore we find that:

$$s(p) = \lim_{N\to\infty} \left(\frac{1}{N} \ln C\right) = \lim_{N\to\infty} \frac{1}{N} (\ln C_0 + \ln C_\infty). \tag{11}$$

But the 'other' sites F_0 and E_0 on the lattice are comprised of fractions $p(F_0)$ and $q(E_0)$ with the ratio derived in the proof of (i). It then follows that the specific logarithmic multiplicity of configurations C_0 is given by equation (8) and that

$$\lim_{N\to\infty} \left(\frac{1}{N} \ln C_0\right) = (p(F_0) + q(E_0)) s(p). \tag{12}$$

Now using the fact that $1 - p(F_0) - q(E_0) = p(F_c) + q(E_b)$ from equations (4*a*) and (4*b*), equation (11) then yields:

$$(p(F_c) + q(E_b))^{-1} \lim_{N\to\infty} \left(\frac{1}{N} \ln C_\infty\right) = s(p). \tag{13}$$

But the left-hand side of equation (13) is precisely the specific logarithmic multiplicity of all infinite clusters per occupied site and defining boundary site. Since $p(F_c)/(p(F_c) + q(E_b)) = p$ as in the proof of (i), it follows that the specific logarithmic multiplicity of all infinite clusters per occupied cluster site is given by:

$$\frac{1}{p(F_c)} \lim_{N\to\infty} \left(\frac{1}{N} \ln C_\infty\right) = S(p) = \ln p + \frac{1-p}{p} \ln(1-p). \tag{14}$$

But this is the same as equation (9) and property (ii) is proven.

Property (iii)

This follows very simply by substituting equation (4) in equation (14) yielding equation (2).

We now give a discussion of the significance of these results.

Evidence for equation (4) has recently been obtained empirically† from Monte Carlo methods on a two-dimensional lattice by Stoll and Domb (1978). It would be

† I am grateful to Professor Domb for bringing this to my attention.

most interesting to test this relationship on higher-dimensional lattices by similar methods.

Equation (4) has interesting applications to Ising models since for $T > T_c$, $(1-p)/p$ is strictly equal to $e^{-Hs/kT}$ where H is the magnetic field. In the non-interacting limit $J/T \to 0$ spin direction is strictly random and the entropy is given by equation (8) whilst equation (4) may be interpreted as giving the ratio a for the infinite cluster(s). For spins parallel to the field $a = e^{-H/kT}$ and for antiparallel spins $a = e^{+H/kT}$: such infinite clusters cannot exist for a less than the percolation limit a_c, giving limiting values to H: $H_c = kT \ln a_c$. If $p_c > \frac{1}{2}$ then $a_c < 1$ and at $H = 0$ neither infinite cluster exists. For lattices on which $a_c > 1$ a limited range of H allows coexisting infinite clusters of opposite spins.

As Domb (1976) has pointed out, the behaviour at large n of $M(n, b)$ for b/n fixed is of great interest to percolation theory. Approximations to $[\ln M(n, b)]/n$ may be simply derived from equation (2) since the infinite cluster has a well defined density of sites $p(F_c)$ and a well defined value of a to which hypercubes (or other suitable hypervolumes) of lattice will approximate arbitrarily closely if of large enough size. Let us divide the partially filled lattice into hypercubes of a given size; such a set of hypercubes will contain a sufficient multiplicity of cluster shapes C to yield equation (2) when these are summed over the entire lattice. If we assume a dominance of the mean values $\bar{n}$ and $\bar{b}$ we automatically find that the multiplicity $M_c(\bar{n}, \bar{b})$ of hypercubes containing $\bar{n}$ infinite cluster sites and $\bar{b}$ defining boundary sites is given by

$$\frac{1}{\bar{n}} \ln M_c(\bar{n}, \bar{b}) = S(a) \tag{15}$$

and hence we can evaluate the number of possible clusters *contained within* the defining volume of a hypercube and having n sites and b defining boundary sites as

$$\frac{1}{n} \ln M_c(n, b) = S(a') \tag{16}$$

where

$$a' = \frac{b}{n} - \delta_c a \tag{17}$$

and $\delta_c a$ is a small increase in the value of a to a' caused by the *new* boundary sites at the surface of the hypercube where there were full infinite cluster sites in adjoining hypercubes previously.

Let us now generalise equation (16). If some other shapes of cluster containing n sites and b boundary sites were taken from the infinite cluster one would expect its multiplicity to be given by an analogous equation

$$\frac{1}{n} \ln M_s(n, b) = S(a - \delta_s a). \tag{18}$$

Equation (18) is proposed for $a < a_c$ and $S(a)$ given by equation (2). Since equation (2) gives $S(a)$ as an *increasing* function of a it is clear that the maximum value of $\ln M_s(n, b)$ will be gained by taking a shape which minimises the small increase in $\delta_s a$. Such shapes must obviously minimise the hypersurface area to hypervolume ratio and are hyperspheres.

To evaluate this completely it is necessary to perturb suitably about the minimum $\delta_s a$ and to sum over all such perturbations to include all possible external shapes of finite cluster. For sufficiently large n the dominance of the hyperspheres gives rise to the following results:

$$S_n(a)=\frac{1}{n}\ln M(n,b)|_{b/n=a}$$

$$\simeq(1+a)\ln(1+a)-a\ln a-n^{-1/d}A(a,d)\ln\left(\frac{1+a}{a}\right) \quad (19)$$

where $A(a,d)$ is only a function of the ratio a, the lattice dimensionality d and *not* of n; it is approximately given by its values for a hypersphere for which

$$A(a,d)=A_{\rm HS}(a,d)=f\left(\frac{d}{p(F_c)}\right)^{1-(1/d)}S_d^{1/d} \quad (20)$$

where S_d is the surface area of a hypersphere of unit radius in d dimensions, $p(F_c)$ is defined as before and f is the fraction of 'created boundary sites' caused by cutting the infinite cluster at a given value of p: both depend on a and d. We understand that the $n^{-1/d}$ relationship has recently been derived independently (Kunz, Stauffer preprints).

As the critical point a_c is approached, singularities in $A(a,d)$ will take effect on account of the singular behaviour of both $p(F_c)$ and f, and these will contribute to the critical exponents.

The fact that $S_n(a)$ tends to a well defined limit which is the corresponding quantity for the infinite cluster may be used to provide a simple analytic proof of the main results of this paper. Consider the probability $P(n,b)$ that any site be a member of an (n,b) cluster

$$P(n,b)=M(n,b)p^n(1-p)^b. \quad (21)$$

Taking logarithms and keeping b/n fixed and equal to a, we have

$$\ln P(n,a)=n[S_n(a)+\ln p+a\ln(1-p)]. \quad (22)$$

But $\ln P(n,a)$ must be 0 or less for all n; hence we have the inequality

$$S_n(a)\leq -\ln p-a\ln(1-p); \quad (23)$$

or taking the maximum value of the right-hand side as a function of p yields equation (4) and

$$S_n(a)\leq(1+a)\ln(1+a)-a\ln a. \quad (24)$$

Let us suppose for example that

$$S_n(a)\simeq S(a)-n^{-\sigma}f_{\rm L}(a)$$

where $f_{\rm L}(a)$ is some lattice-dependent function. We know that for $a<a_c$ an infinite cluster occurs with finite probability $p(F_c)$. Hence the limit

$$\lim_{n\to\infty}\ln P(n,a)=\lim_{n\to\infty}\{n[S(a)+\ln p+a\ln(1-p)]-n^{1-\sigma}f_{\rm L}(a)\}$$

is strictly divergent but yields equations (2) and (4) as *necessary* conditions for any class of infinite cluster to be realised by random occupation of any lattice of finite dimensionality. Therefore $S(a)<(1+a)\ln(1+a)-a\ln a$ when $a>a_c$ for which values of a infinite clusters do *not* occur spontaneously. Hence we suggest that at $a = a_c$, $S(a)$ is not analytic, and that $S(a)$ consists of two parts:

(i) $(a < a_c)$ for which $S(a)$ is given by equation (2);

(ii) $(a > a_c)$ for which $S(a)$ is a lattice-dependent function.

If $S(a)$ could be evaluated exactly on a particular lattice for $a > a_c$ the value of a_c (and hence of p_c) would become known. Preliminary work indicates that for $a > a_c$, $\mathrm{d}S/\mathrm{d}a \leq 0$ and that $S(a_c) = \max S(a)$, whilst the non-analyticity of $S(a_c)$ is caused by a discontinuity in $\mathrm{d}S/\mathrm{d}a$. If this is true, then $S(a_c)$ is indeed precisely the value Λ which controls the maximum possible number of clusters for given n as n becomes infinite—a very elegant result previously suggested by Domb (1976).

The results in this paper came intuitively whilst participating in the AEGTC at MERU in 1977. I should like to express my great gratitude to Maharishi Mahesh Yogi and the faculty of MERU for making this programme and its profound results available to me.

I should like to thank Professor H E Stanley for bringing the problem of percolation to my attention. Finally I am deeply indebted to Professor C Domb for his kind comments and criticisms of earlier versions of this work and I am particularly grateful to him for his warmth and encouragement.

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